WHY DOES PLATO'S ELEMENT THEORY CONFLICT WITH MATHEMATICS (ARIST. CAEL. 299a2–6)?

In Cael. 3.1 Aristotle argues against those who posit that all bodies are generated because they are made from, and dissolve into, planes, namely Plato and perhaps other members of the Academy who subscribed to the Timaeus physics (cf. Simplicius, In Cael. 561,8–11 [Heiberg]). In his *Timaeus* Plato assigns to each of the traditional Empedoclean elements a regular polyhedron: the tetrahedron or pyramid to fire, the cube to earth, the octahedron to air and the icosahedron to water. Each regular polyhedron can be anachronistically called a molecule of the element in question and, as is suggested by the analogy between the regular solids and molecules, Plato also posits that the regular polyhedra are made from 'atoms': the faces of the tetrahedron, octahedron and icosahedron are made from scalene right-angled triangles, whose hypotenuses are double the length of the smaller sides, whereas the faces of the cube consist of isosceles right-angled triangles. Since fire, air and water consist of polyhedral molecules whose elementary constituents are of the same type, they can freely change into one another. Any of these three elements turns into another when its molecules break down into their elementary constituents and these building blocks recombine into molecules of another element[1]. Aristotle has in mind the reshuffling of elementary triangles when he refers to all bodies being made from, and dissolving into, planes. His first objection to this fundamental assumption in Plato's element theory is set out in Cael. 299a2–6: as is easily seen, constructing bodies from

1) Since, however, the molecules of earth on the one hand and the molecules of fire, air and water on the other consist of elementary components of different types, neither fire nor air nor water can change into earth and similarly earth cannot turn into any of these elements. For illustrative examples of Timaean elemental transformations see G. Vlastos, Plato's Universe (Oxford 1975) 70–72. Cf. L. Brisson & F. W. Meyerstein, Inventing the Universe: Plato's Timaeus, the Big Bang and the Problem of Scientific Knowledge (Albany, NY 1995) 40–49.

planes runs counter to mathematics whose 'hypotheses' should be accepted, unless one comes up with something more convincing[2].

Contrary to Aristotle's claim it is not easy to see why Plato's element theory runs counter to mathematics because it constructs the polyhedral molecules from the triangular planes in the faces of these molecules. Aristotle presumably implies that this violates some mathematical 'hypotheses' which should be better left as they stand but does not explain what the 'hypotheses' in question are. Nor is it any clearer whether Plato commits himself to the rejection of these 'hypotheses' or some aspect of Plato's element theory entails their rejection by Aristotle's own lights. I will attempt to answer these questions after a critique of Simplicius who identifies the hypotheses in Cael. 299a2–6 with the Euclidean definitions of point, line and plane but also thinks that Aristotle sets out further mathematical objections to Plato's element theory in Cael. 299a6–11: contrary to the commentator there is only one such objection in Cael. 299a6–11, namely that Plato's element theory introduces indivisible lines, and, as is suggested by an allusion to Cael. 299a2–6 in the treatise *On Indivisible Lines*, the same objection is also implicit in Cael. 299a2–6; that in this passage Plato's element theory is said to conflict with mathematics because it entails the existence of indivisible lines is borne out not only by Cael. 299a6–11 but also by 299a13–17. After interpreting the 'hypotheses' in Cael. 299a2–6 consistently with this fact, I will show that, when Aristotle charges Plato with introducing various sorts of indivisibles in his element theory, he actually brings out the untenability of this theory by arguing that Plato ought to introduce such entities which are, though, ruled out by mathematics. Aristotle's implicit objection in Cael. 299a2–6 follows from a similar argument which I will attempt to reconstruct in the final sections of this paper.

1. Simplicius' interpretation of Cael. 299a2–6

Simplicius understands the 'hypotheses' of mathematics as the definitions of point, line and plane. According to the commentator

---

2) τοῖς δὲ τοῦτον τὸν τρόπον λέγουσι καὶ πάντα τὰ σώματα συνιστᾶσιν ἐξ ἐπιπέδων ὅσα μὲν ἄλλα συμβαίνει λέγειν ὑπεναντία τοῖς μαθήμασιν, ἐπιπολῆς ἰδεῖν· καίτοι δίκαιον ἦν ἢ μὴ κινεῖν ἢ πιστοτέροις αὐτὰ λόγοις κινεῖν τῶν ὑποθέσεων.

in Cael. 299a2–6 Aristotle tacitly assumes that, if bodies are made from planes, then planes are made from lines and lines from points. He then argues that, since lines are extended, they can be made from points only if points are extended or, equivalently, have parts; similarly, planes can be made from lines and bodies from planes only if lines have breadth and planes have depth, because planes have breadth and bodies have depth so that the putative constituents of planes and bodies must also have breadth and depth respectively; this, however, runs counter to mathematics, for a point by definition lacks parts, a line breadth and a plane depth[3].

The implicit premise Simplicius reads in Cael. 299a2–6 (if bodies are made from planes, then planes are made from lines and lines from points) is Aristotle's immediately following argument in Cael. 299a6–11 where, though, it leads to the further conclusion that the part of a line is not necessarily a line: since Aristotle counters that, as he has shown in the treatise on motion (more on that below), there are no indivisible lines, the part of a line is not necessarily a line in that it is not a divisible line and thus the absurdity Aristotle sees implicit in constructing bodies from planes as Plato does is that the parts of a line are indivisible lines[4]. Commenting on the immediately following lines (Cael. 299a11–13), where Aristotle announces that he will bring out certain physical absurdities implicit in the supposition of indivisible lines, Simplicius points out that, although in Cael. 299a2–6 Aristotle skips the mathematical objections to Plato's element theory, he actually sets out the most important of them in Cael. 299a6–11: Plato's element theory not only conflicts with some fundamental definitions of geometry, as Aristotle hints in Cael. 299a2–6 according to Simplicius, but also entails, contrary to what is as-

3) In Cael. 562.21–30 (Heiberg): Πρῶτον ἔγκλημα τούτοις ἐπάγει τὸ τὰς γεωμετρικὰς ἀρχὰς ἀναιρεῖν πρόχειρον εἶναι λέγων τὴν τούτου κατανόησιν· διὸ καὶ παρῆκεν αὐτήν. λέγει δὲ ἀναιρεῖν αὐτοὺς τοὺς τῶν γεωμετρῶν ὅρους τοῦ τε σημείου καὶ τῆς γραμμῆς καὶ τῆς ἐπιφανείας· εἰ γὰρ σημεῖον λέγουσιν, οὗ μέρος οὐθέν, γραμμὴν δὲ μῆκος ἀπλατές, ἐπιφάνειαν δέ, ὃ μῆκος καὶ πλάτος μόνον ἔχει, οὐκ ἄν ποτε ἐκ σημείων γραμμὴ γένοιτο, ὥστε οὐδὲ ἐκ γραμμῶν ἐπιφάνεια οὐδὲ ἐξ ἐπιφανείας ἀβαθοῦς οὔσης σῶμα βεβαθυσμένον· εἰ δὲ γίνεται ἐξ ἐπιπέδου σῶμα, βάθος ἂν ἔχοι τὸ ἐπίπεδον, καὶ εἰ ἐκ γραμμῶν ἐπίπεδον, οὐκ ἂν εἴη ἀπλατὴς ἡ γραμμή, καὶ εἰ ἐκ σημείων γραμμή, οὐκ ἂν ἀμερὲς εἴη τὸ σημεῖον. Simplicius operates with the Euclidean definitions of point, line and plane (El. 1 Def. 1, 2 and 5 respectively).

4) Cael. 299a6–11: Ἔπειτα δῆλον ὅτι τοῦ αὐτοῦ λόγου ἐστὶ στερεὰ μὲν ἐξ ἐπιπέδων συγκεῖσθαι, ἐπίπεδα δ' ἐκ γραμμῶν, ταύτας δ' ἐκ στιγμῶν· οὕτω δ' ἐχόντων οὐκ ἀνάγκη τὸ τῆς γραμμῆς μέρος γραμμὴν εἶναι· περὶ δὲ τούτων ἐπέσκεπται πρότερον ἐν τοῖς περὶ κινήσεως λόγοις, ὅτι οὐκ ἔστιν ἀδιαίρετα μήκη.

sumed in mathematics, that the part of a line is not a line, that a line consists of points and that magnitudes are not ad infinitum divisible; these objections, Simplicius notes, are also to be found in the treatise *On Indivisible Lines* which some attribute to Theophrastus[5].

2. The number of objections to Plato's element theory in Cael. 299a6–11

Irrespective of whether this Peripatetic treatise is to be ascribed to Theophrastus and is directed against Plato's element theory as Simplicius seems to assume, there is no reason to read with the commentator three implicit mathematical objections to this theory in Cael. 299a6–11. Aristotle claims that the fundamental assumption in Plato's element theory (the composition of elementary bodies from elementary planes) entails the composition of lines from points, which Simplicius apparently takes to be in itself damaging to Plato's element theory from a mathematical point of view. However, the composition of lines from points entails in its turn that the part of a line is not necessarily a line and, contrary to Simplicius, there can be no difference between a part of a line being not necessarily a line and, in general, magnitudes such as lines being not ad infinitum divisible: for, as seen above, that the part of a line is not necessarily a line means that it is an indivisible line or, equivalently, that lines are not ad infinitum divisible (since they consist of indivisible lines). Thus according to Simplicius there can be at most two mathematical objections to Plato's element theory implicit in Cael. 299a6–11: contrary to what is entailed by the fundamental assumption in this theory, lines cannot consist of points and there are no indivisible lines – as Aristotle puts it in Cael. 271b9–11, by introducing a minimum and thus indivisible magnitude one shakes τὰ μέγιστα τῶν μαθηματικῶν.

5) Simpl. In Cael. 566.23–567.1 (Heiberg): Τὰς μὲν ἀπὸ τῶν μαθημάτων ὁρμωμένας ἐνστάσεις πρὸς τοὺς ἐξ ἐπιπέδων τὰ σώματα γεννῶντας, ὡς μὲν ἄν τῳ δόξειεν, ὑπερέθετο νῦν καὶ ὡς προχείρους ἰδεῖν καὶ ὡς ἐν τῷ Περὶ τῶν ἀτόμων γραμμῶν περὶ αὐτῶν εἰρηκώς, ὅ τινες εἰς Θεόφραστον ἀναφέρουσιν, ὡς δὲ τὸ ἀληθὲς ἔχει, κατὰ τὸ παραλειπτικὸν παρὰ τοῖς ῥήτορσι καλούμενον σχῆμα καὶ τούτων τὰς κυριωτέρας παρήγαγεν, ὅτι ἀναιρεθήσονται αἱ ὁριστικαὶ τῶν μαθημάτων ἀρχαί, ὅτι τὸ τῆς γραμμῆς μέρος οὐκ ἔσται γραμμή, ὅτι ἡ γραμμὴ ἐκ στιγμῶν ἔσται συγκειμένη, ὅτι τὰ μεγέθη οὐκ ἔσται ἐπ' ἄπειρον διαιρετά.

If, though, one takes one's cue from the treatise *On Indivisible Lines* as Simplicius does, these two objections are actually one and the same. The anonymous author notes that there is no difference between points and indivisible lines (970b29–30)[6] and that most of the arguments against the hypothesis of indivisible lines, some arguments from mathematics included, also rule out the composition of lines from points (971a3–7)[7]. The hypotheses that lines are composed from indivisible lines and that they consist of points are, therefore, equivalent. If Aristotle takes tacitly this equivalence for granted, one can easily explain, why he concludes in Cael. 299a6–11 that Plato is committed to the existence of indivisible lines because – on the fundamental assumption of Plato's element theory – lines turn out to consist of points. Apart from this, however, it stands to reason that, contrary to Simplicius, in Cael. 299a6–11 Aristotle can raise at most one mathematical objection to Plato's element theory different from that implicit in Cael. 299a2–6: the fundamental assumption in Plato's element theory entails the existence of indivisible lines and in that respect it inexorably clashes with mathematics.

3. The mathematical objection to Plato's element theory in Cael. 299a2–6

The author of the treatise *On Indivisible Lines*, however, does not seem to have read a different mathematical objection to Plato's element theory in Cael. 299a2–6 as Simplicius does. There is a clear parallel between the wording of this passage and the introduction to the mathematical arguments against the hypothesis of indivisible lines in the Peripatetic treatise: this assumption is ruled out πρῶτον μὲν ἐκ τῶν ἐν τοῖς μαθήμασι δεικνυμένων καὶ τιθεμένων, ἃ δίκαιον ἢ μένειν ἢ πιστοτέροις λόγοις κινεῖν (969b29–31)[8] which resembles

6) οὐδὲν γὰρ ἴδιον ἕξει ἡ ἄτομος γραμμὴ παρὰ τὴν στιγμὴν πλὴν τοὔνομα.

7) See the brief comments in H. H. Joachim, De Lineis Insecabilibus (Oxford 1908) ad loc. and M. Timpanaro Cardini, Pseudo-Aristotele: De lineis insecabilibus (Milan 1970) 94. Although points are indivisible, they cannot be self-evidently treated as if they were indivisible lines. Unlike indivisible lines, points by definition lack extension and there is no reason why the former should behave like the latter.

8) I cite the text of O. Apelt, Aristotelis quae feruntur De plantis, De mirabilibus auscultationibus, Mechanica, De lineis insecabilibus, Ventorum situs et nomina, De Melisso Xenophane Gorgia (Leipzig 1888).

καίτοι δίκαιον ἢ μὴ κινεῖν ἢ πιστοτέροις αὐτὰ λόγοις κινεῖν τῶν ὑποθέσεων in Cael. 299a5–6[9]. If this similarity is not accidental, it suggests that the Peripatetic author wrote with Cael. 299a2–6 in mind and read in this passage an allusion to indivisible lines: according to the anonymous author the fundamental assumption in Plato's element theory is said in Cael. 299a2–6 to clash with mathematics in that it entails the existence of indivisible lines. If, therefore, Cael. 299a2–6 and 299a6–11 are read in the light of the treatise *On Indivisible Lines* as Simplicius suggests, one must conclude that in these passages Aristotle claims that Plato's element theory conflicts with mathematics not on four counts as Simplicius takes it but on one and the same count, namely in that it entails the existence of indivisible lines.

4. The physical character of the objections to Plato's element theory in Cael. 299a6–11

This is also suggested by the fact that a few lines after Cael. 299a2–6 Aristotle hints once more at mathematical difficulties that bedevil Plato's element theory. On this occasion, however, these difficulties clearly arise from the existence of indivisible lines Aristotle sees implicit in the fundamental assumption of this theory: there is, consequently, no reason to follow Simplicius in assuming that in Cael. 299a2–6 Aristotle hints at some other mathematical difficulties with the theory and, what is more, the context leaves no doubt that pace Simplicius in Cael. 299a6–11 Aristotle has nothing to say on why in Cael. 299a2–6 he claims that Plato's element theory clashes with mathematics. As seen above, in Cael. 299a6–11 Aristotle notes that the composition of lines from points and, equivalently, the existence of indivisible lines has been ruled out in the treatise on motion. This treatise must contain an argument to the effect that there are no indivisible lines and thus it can only be Phys. Z where Aristotle infers indirectly the divisibility of any line by showing that a line or, in general, a continuum cannot consist of points or, in general, indivisibles (Phys. 231a21–232a22). Since,

9) This similarity is noted in the apparatus criticus in Paul Moraux, Aristote: Du Ciel (Paris 1965) 105 and Timpanaro Cardini (above, n. 7) 56; see also Joachim (above, n. 7) on 969b30–31.

however, immediately after his reference to the treatise on motion Aristotle goes on to announce that on this occasion too he will be concerned with the physical absurdities facing those who like Plato posit indivisible lines (Cael. 299a11–13)[10], he can only take for granted that his argument in the treatise on motion has exposed physical absurdities implicit in the hypothesis of indivisible lines.

In Cael. 299a13–17 Aristotle proceeds to distinguish between two kinds of absurdities, physical and mathematical, facing those who posit indivisible lines and calls attention to an interesting asymmetry: the mathematical absurdities in question are also physical whereas not all physical absurdities are necessarily mathematical[11]. Thus on Cael. 299a6–11 the fundamental assumption in Plato's element theory entails the composition of lines from points and, consequently, the existence of indivisible lines (provided of course that points and indivisible lines are tacitly conflated as is the case in the treatise *On Indivisible Lines*) but on Cael. 299a13–17 the assumption of indivisible lines gives rise to mathematical as well as purely physical absurdities. Since the latter are ferreted out in Cael. 299a17–b14, it stands to reason that the concomitant mathematical absurdities are alluded to in Cael. 299a2–6 where Aristotle points out in passing that Plato's element theory conflicts with mathematics. If, moreover, Aristotle intends Cael. 299a6–11 to ground implicitly his claim in Cael. 299a2–6 that Plato's element theory clashes with mathematics as Simplicius takes it, he must understand any absurdities he derives in Phys. Z from the hypothesis that lines consist of points not as purely physical and, therefore, as mathematical. This is, though, evidently not the case, for in Phys. Z Aristotle leads the composition of continua from points to purely kinematic absurdities and it is not accidental that in GC 316a15–34 the composition from points is rejected not as a mathematical but as a physical absurdity. That for Aristotle his treatise on motion renders untenable the assumption of indivisibles by laying bare purely physical absurdities implicit in this assumption is shown by Cael. 303a20–24 where Aristotle points out that Democritus' hypothesis of atoms conflicts, first, with mathematics and, second, with πολλὰ τῶν ἐνδόξων καὶ τῶν φαινομένων κατὰ τὴν αἴσθησιν, as has already been shown in the treatise on time and motion.

10) Ὅσα δὲ περὶ τῶν φυσικῶν σωμάτων ἀδύνατα συμβαίνει λέγειν τοῖς ποιοῦσι τὰς ἀτόμους γραμμάς, ἐπὶ μικρὸν θεωρήσωμεν καὶ νῦν.

5. The mathematical 'hypotheses' in Cael. 299a2–6

In light of the above Cael. 299a2–6 should be fleshed out as follows: Plato's element theory runs counter to certain mathematical 'hypotheses' because it constructs the fundamental physical bodies (the polyhedral molecules of the four elements) from planes, which entails the existence of indivisible lines – it is thus indivisible lines which ultimately clash with the 'hypotheses' of mathematics in Cael. 299a2–6. In An. Post. 72a14–24 the 'hypotheses' are defined as assumptions of existence[12] so that the existence of indivisible lines must conflict with some 'hypotheses' of mathematics in that, if there are such lines, the mathematical objects whose existence is asserted by the 'hypotheses' in question cannot obtain. Now one of the mathematical arguments against indivisible lines in the treatise *On Indivisible Lines* is that on the assumption of such lines there are neither incommensurable lines nor irrational squares (969b33–970a4) and, since for Aristotle definitions entail existential propositions (An. Post. 92b4–11), the 'hypotheses' in Cael. 299a2–6 might very well be the existential assumptions entailed by the definitions of incommensurable lines and irrational squares current in the mathematics of Aristotle's time (cf. Euclid, El. 10 Def. 1, 4). In EE 1227b23–32, however, Aristotle uses the term 'hypothesis' to denote not existential assumptions entailed by definitions but any proposition which is taken for granted and is used in a proof[13]: if this is the sense of 'hypothesis' in Cael. 299a2–6, the 'hypotheses' which the assumption

---

11) The reason for this is that mathematicals are abstracted from physical objects. Solids, planes, lines and points or, in general, the objects mathematics studies are in physical bodies (Phys. 193b22–25) but mathematics does not study the properties of these objects qua physical. It disregards, or abstracts from, those physical properties (e. g. motion) which are irrelevant to its study, in other words it isolates conceptually only certain properties of physical objects (i. e. the properties of these objects qua quantities and continua in one, two or three dimensions; Met. 1061a; 1077b22–32); see Phys. 193b25–35 and J. Lear, Aristotle's Philosophy of Mathematics, in: L. P. Gerson (ed.), Aristotle: Critical Assessments. Vol. 1 Logic and Metaphysics (London & New York 1999) 141–166 (= PhR 91, 1982, 161–192). Thus the assumption of indivisible lines gives rise to mathematical absurdities when checked against the properties of physical objects studied by mathematics but also to purely physical absurdities when checked against those properties of physical objects that fall outside the scope of mathematics.

12) See J. Barnes, Aristotle: Posterior Analytics (Oxford [2]1994) 100.

13) See Barnes (above, n. 12) 100.

of indivisible lines conflicts with must be certain propositions of a rather elementary nature like the well-known construction in Euclid's El. 1.10 (to bisect a given finite straight line)[14] or the ancestor of Euclid's El. 10.1 (Aristotle refers to the ancestor of El. 10.1 in Phys. 266b2–4: any given magnitude can be exceeded by means of continual subtraction from another magnitude)[15].

6. Does the Timaeus element theory entail the existence of indivisibles?

There is nothing in Plato's *Timaeus* to suggest that constructing the fundamental bodies from triangular planes entails the existence of indivisible lines; Aristotle's report that Plato used to reject points as geometrical fictions and posited instead indivisible lines as the principle of lines (Met. 992a20–21) does not mention the Timaeus element theory as the context of Plato's enigmatic move[16]. That, as a consequence, the assumption of indivisible lines is implicit in Plato's element theory only by Aristotle's lights can also be concluded from Cael. 306a26–30 where Aristotle points out once again that Plato's element theory conflicts with mathematics but for reasons different from those in Cael. 299a2–6, namely because in Plato's element theory there are indivisible bodies[17]. The

14) According to the treatise *On Indivisible Lines* the existence of indivisible lines is ruled out by τὰ ἐν τοῖς μαθήμασι δεικνύμενα (see 969b29–31), which can only be certain theorems or problems like El. 1.10. On El. 1.10 as ruling out indivisible lines see Proclus, In Pr. El. Comm. 277.25–279.4 (Friedlein).

15) See J. L. Heiberg, Mathematisches zu Aristoteles, Abhandlungen zur Geschichte der Mathematischen Wissenschaften 18, 1904, 23 and W. R. Knorr, Archimedes and the Pre-Euclidean Proportion Theory, AIHS 28, 1978, 210. According to the treatise *On Indivisible Lines* the hypothesis of indivisible lines is to be rejected on account of its incompatibility with the definitions of the line and the straight line (969b31–33), which are characterized as τὰ ἐν τοῖς μαθήμασι τιθέμενα (see 969b29–31). The definitions referred to are presumably 'that which is between two points' (cf. Aristotle, Phys. 231b9) and 'that whose middle point is in the way of both ends' (cf. Plato, Parm. 137e2–3); see Joachim (above, n. 7) on 969b31–33 and Timpanaro Cardini (above, n. 7) 86.

16) Nor is it possible to understand the reasons for this move; cf. J. Annas, Aristotle's Metaphysics: Books M and N (Oxford 1976) 25.

17) Πρὸς δὲ τούτοις ἀνάγκη μὴ πᾶν σῶμα λέγειν διαιρετόν, ἀλλὰ μάχεσθαι ταῖς ἀκριβεστάταις ἐπιστήμαις· αἱ μὲν γὰρ καὶ τὸ νοητὸν λαμβάνουσι διαιρετόν, αἱ μαθηματικαί, οἱ δὲ οὐδὲ τὸ αἰσθητὸν ἅπαν συγχωροῦσι διὰ τὸ βούλεσθαι σῴζειν τὴν ὑπόθεσιν.

context leaves no doubt that the indivisible bodies Aristotle has in mind here are the polyhedral molecules of the four elements but elsewhere he reports that the indivisible magnitudes in Plato's element theory are neither lines nor solids but planes, namely the triangles that make up the polyhedral molecules of the four elements: in GC 315b24–32 Aristotle distinguishes the indivisible bodies of Democritus and Leucippus from the indivisible planes Plato introduces in his *Timaeus* and remarks that atomic bodies are preferable to atomic planes[18]. Since in GC 315b24–32 the indivisible magnitudes attributed to Plato are clearly not bodies but the triangles in the faces of the Timaeus polyhedral molecules, Aristotle must assert the indivisibility of these molecules in Cael. 306a26–30 on purely polemical grounds (were Plato committed to the indivisibility of his polyhedral molecules, in GC 315b24–32 Aristotle would not specify that of those who posit indivisible magnitudes some, like Leucippus and Democritus, posit indivisible bodies but others, like Plato, opt for indivisible planes).

Plato is indeed clear that the triangles in the faces of the polyhedral molecules are parts of these solids (Ti. 56d4) and that in their interactions with molecules of other elements the molecules of one element are divided into their constituent parts (Ti. 56d6.e6; 57b1)[19]; thus the polyhedral molecules of the four elements cannot be indivisible, either physically or conceptually[20]. In Cael. 306a26–30, however, Aristotle argues that, if the polyhedral molecule of e. g. fire is divisible like any mathematical solid, its parts cannot be parts

18) Ἀρχὴ δὲ τούτων πάντων, πότερον οὕτω γίνεται καὶ ἀλλοιοῦται καὶ αὐξάνεται τὰ ὄντα καὶ τἀναντία τούτοις πάσχει, τῶν πρώτων ὑπαρχόντων μεγεθῶν ἀδιαιρέτων, ἢ οὐθέν ἐστι μέγεθος ἀδιαίρετον· διαφέρει γὰρ τοῦτο πλεῖστον. Καὶ πάλιν εἰ μεγέθη, πότερον, ὡς Δημόκριτος καὶ Λεύκιππος, σώματα ταῦτ' ἐστίν, ἢ ὥσπερ ἐν τῷ Τιμαίῳ ἐπίπεδα. Τοῦτο μὲν οὖν αὐτό, καθάπερ καὶ ἐν ἄλλοις εἰρήκαμεν, ἄλογον μέχρι ἐπιπέδων διαλῦσαι. Διὸ μᾶλλον εὔλογον σώματα εἶναι ἀδιαίρετα.

19) There is no reason to assume that in Cael. 306a26–30 the Timaeus polyhedral molecules are indivisible in the sense that, although they do have two-dimensional parts, they cannot be divided into solids. Nowhere does Plato suggest that this is so and, if the molecules of the four elements are regular solids enclosing empty space (see R. D. Mohr, The Platonic Cosmology [Leiden 1985] 112–115), there is no reason why on some occasions they cannot be broken into solid fragments which will become parts of a molecule again when they encounter complementing molecular fragments or a sufficient number of free elementary triangles of the right type.

20) For the distinction between physical and conceptual indivisibility see R. Sorabji, Time, Creation and the Continuum (London 1983) 352.

of fire and there is something prior to fire, apparently the parts into which the molecule of fire can be divided, because every physical body is either an element or made from elements (see Cael. 306a30–b2). Aristotle's point is that, since the molecule of fire must be made from the parts into which it can be divided, these parts can only be the elements of the fire molecule: thus, if one is to rule out the existence of elements of the elements, the polyhedral molecules of the four elements must be indivisible unlike any mathematical solid. Far from presupposing that Plato assumes the indivisibility of the polyhedra he treats as molecules of fire or any other element, Aristotle argues that Plato ought to take this indivisibility for granted, for the polyhedral molecules cannot behave like mathematical solids which are divisible into two or more smaller solids[21]. Thus by Aristotle's lights Plato necessarily stumbles upon the mathematical absurdity of positing indivisible solids if he tries to avoid the physical absurdity implicit in the natural assumption that the polyhedral molecules are divisible like mathematical solids.

Since in Cael. 306a26–30 Aristotle argues that the polyhedral molecules of the four elements must be indivisible although there is no evidence in Plato's *Timaeus* as to their indivisibility, one is alert to the possibility that in GC 315b24–32 the indivisibility of the elementary triangles in the faces of these molecules is also foisted on Plato by Aristotle. In GC 315b24–32 Aristotle has clearly the conceptual indivisibility of the elementary triangles in mind because he explains their indivisibility as if each elementary triangle were not a physical entity but a Platonic idea – if these triangles were divisible, the triangle-in-itself would be divisible (GC 316a10–14). There can be no doubt that, unlike the polyhedral molecules, the elementary triangles must be physically indivisible because nowhere does Plato hint at their fission, but he does not seem to hint at their conceptual indivisibility either and there is no reason to

21) In the treatise *On Indivisible Lines* (968a14–18) Aristotle's reasoning turns into an argument in favor of indivisible physical bodies (if there are elements of compound bodies and there is nothing prior to the elements but the parts are prior to the whole, fire and the other three elements of compound bodies must be indivisible so there are indivisibles among not only intelligible but also physical objects). This argument can be plausibly attributed to Xenocrates who seems to have followed Plato in associating a regular polyhedron with each of the four elements (fr. 53 Heinze) and is reported to have posited the composition of the four elements from elementary minimal solids (fr. 51 Heinze); cf. R. Heinze, Xenokrates (Leipzig 1892) 68–69.

think that their conceptual indivisibility is entailed by some aspect of his element theory[22]. That Plato would not posit the conceptual indivisibility of his elementary triangles is perhaps suggested by his argument in *Parmenides* (137d4–139b2) that anything lacking parts can have neither shape nor location and can neither move nor rest, though it is admittedly difficult to decide whether Plato is committed to any argument he sets out in the second part of this dialogue. For Aristotle, however, the assumption of indivisible physical objects like Plato's elementary triangles entails the supposition of conceptually indivisible mathematical objects as is shown by his argument implicit in Cael. 303a20–24 to the effect that Democritus' atoms, which can only be physically indivisible solids, are ruled out by mathematics: if there are physically indivisible solids, as Democritus assumes, there are conceptually indivisible mathematical solids but, since all mathematical solids are by definition divisible, there can be no physically indivisible solids[23]. Similarly, if there are physically indivisible planes, as Plato assumes, there are also conceptually indivisible mathematical planes.

7. The grounds for Aristotle's view that Plato's element theory introduces indivisible lines

In Cael. 299a6–11 now the implicit conclusion that the fundamental assumption of Plato's element theory entails the mathematically and physically inadmissible existence of indivisible lines follows from Aristotle's argument that, if bodies are made from planes as Plato assumes, then planes are made from lines and lines from points: as argued above, Aristotle can infer from this that on Plato's element theory there are indivisible lines if he sees no difference between indivisible lines and points, as is also the case in the Peripatetic treatise *On Indivisible Lines*. This conflation is suggested by the context of Cael. 299a6–11. In Cael. 299a11–13 Aris-

22) Against the conceptual indivisibility of Plato's elementary triangles see Sorabji (above, n. 20) 358–359.

23) The implicit assumption spearheading Aristotle's argument is that, as Lear (above, n. 11) 162 puts it, "mathematics is true, not in virtue of the existence of separated mathematical objects to which its terms refer, but because it accurately describes the structural properties and relations which actual physical objects do have"; see also above, n. 11.

totle announces that he will proceed to demonstrate some physical absurdities arising from the supposition of indivisible lines, namely that indivisible entities can only lack certain physical properties (Cael. 299a13–25), and he explains the upshot of this with respect to the property of weight as follows: if what has weight cannot be made from weightless parts and points cannot have weight, neither lines nor planes nor bodies have weight, which is obviously absurd because, as Plato himself would admit, all or some physical bodies do have weight[24]. Aristotle here relies on his conclusion in Cael. 299a6–11 as a premise – if bodies are made from planes as Plato assumes, then planes are made from lines and lines from points (the premise) so that bodies ultimately consist of points and cannot have weight if points are shown not to have weight. Although one would expect, in the light of the programmatic statement in Cael. 299a11–13, that indivisible lines are the entities Aristotle will show to lack weight, what he will show to lack weight turn out to be points and in Cael. 299a30–b11 he indeed argues that points cannot have weight. The unexpected shift of Aristotle's attention from indivisible lines to points is plausibly explained if he does not countenance any difference between points and indivisible lines[25].

Why, though, does Aristotle argue in the first place that, if bodies are made from planes, then by the same reasoning planes are made from lines and, consequently, lines consist of points? That on Plato's element theory lines consist of points is also implicit in Cael. 300a7–12 where Aristotle argues that, if points are to lines as lines are to planes and planes to bodies, each of these entities will dissolve into the other and thus all will dissolve into what is primary so that there can exist no bodies but only points: since he characterizes the elements as 'primary' (Met. 1014a26–27), the term he uses for points in Cael. 300a7–12, and he can only assume that an entity dissolves into what this entity is made from, in Cael. 300a7–12 points are implicitly conceived as elements, i. e. primary irreducible constituents not only of lines but also of bodies. Generally speaking, from the fundamental assumption in Plato's element theory, i. e. that the polyhedral molecules of the four

24) Cael. 299a25–30: Εἰ δὴ τῶν ἀδυνάτων ἐστὶν ἑκατέρου μέρους μηδὲν ἔχοντος βάρος τὰ ἄμφω ἔχειν βάρος, τὰ δ' αἰσθητὰ σώματα ἢ πάντα ἢ ἔνια βάρος ἔχει, οἷον ἡ γῆ καὶ τὸ ὕδωρ, ὡς κἂν αὐτοὶ φαῖεν, εἰ ἡ στιγμὴ μηδὲν ἔχει βάρος, δῆλον ὅτι οὐδ' αἱ γραμμαί, εἰ δὲ μὴ αὗται, οὐδὲ τὰ ἐπίπεδα· ὥστ' οὐδὲ τῶν σωμάτων οὐθέν.

25) Cf. Heinze (above, n. 21) 63 n. 2.

elements or the fundamental bodies are made from and thus dissolve into planes, Aristotle concludes in Cael. 300a7–12 that by Plato's lights planes and lines are made from and thus dissolve into lines and points respectively: not only, therefore, lines turn out to be made from points as is also assumed in Cael. 299a6–11 but the fundamental bodies themselves turn out to ultimately consist not of elementary planes as Plato takes it but of elementary points into which these bodies must necessarily dissolve. That a body consists of points is implicit in Aristotle's argument in Met. 1076a38–b12 against the unorthodox Platonist view that mathematicals are separate from, but in, physical objects[26]. Aristotle argues that, if physical objects are divisible, the separate mathematicals in them must also be divisible: this is impossible because solids can only be divided along planes, planes along lines and lines at points but it is impossible for indivisible points to be divided (since, therefore, physical bodies are divisible, it follows that separate mathematicals cannot be in physical bodies)[27].

This compressed argument can be plausibly fleshed out with the help of another argument of Aristotle's (Phys. 241a15–23) which also entails that an indivisible point turns out to be absurdly divisible on the hypothesis to be refuted. If a point traverses a line, there must be a time less than the time in which the point moves over a part of a line equal to itself: however, in the shorter time the moving point must traverse a part of the line shorter than itself, which is impossible unless the indivisible is divisible because Aristotle has shown that the line which a moving point traverses must consist of points 'equal' to the moving point (Phys. 241a6–14). That a continuum consists of indivisible points here, entails that a point must be absurdly divisible on the hypothesis to be refuted and it stands to reason that this entailment is implicitly at work in Met. 1076a38–b12 as well: if separate mathematical solids

26) For the proponents of this view see Th. Kouremenos, Aristotle on Mathematical Infinity, Stuttgart 1995 (Palingenesia 58), 100–101.

27) Ὅτι μὲν τοίνυν ἔν γε τοῖς αἰσθητοῖς ἀδύνατον εἶναι καὶ ἅμα πλασματίας ὁ λόγος, εἴρηται μὲν καὶ ἐν τοῖς διαπορήμασιν … ἀλλὰ πρὸς τούτοις φανερὸν ὅτι ἀδύνατον διαιρεθῆναι ὁτιοῦν σῶμα· κατ' ἐπίπεδον γὰρ διαιρεθήσεται, καὶ τοῦτο κατὰ γραμμὴν καὶ αὕτη κατὰ στιγμήν, ὥστ' εἰ τὴν στιγμὴν διελεῖν ἀδύνατον, καὶ τὴν γραμμήν, εἰ δὲ ταύτην, καὶ τἆλλα. τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις; τὸ αὐτὸ γὰρ συμβήσεται· διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί.

are in physical bodies, the divisibility of the latter entails absurdly the divisibility of indivisible points because (at least in Aristotle's view) on the unorthodox Platonist theory under attack a mathematical solid consists ultimately of points. Aristotle can reach this conclusion from his thesis that a solid, a plane or a line is divisible along a plane, a line or at a point respectively which are made actual by the division[28]. Assuming that a Platonist is committed to the actuality of all potential planes, lines and points in a solid[29], he can conclude that on the Platonist theory to be refuted lines, planes and thus solids, first, consist ultimately of points and, second, are divisible only if points are absurdly divisible, for points are the only things to be divided. Fair or not, Aristotle's argument seems to presuppose a conception of a solid as a 'stack' of planes, of each plane as a 'bundle' of all lines through it and similarly of each line as a 'string' of points as is the case in Cael. 300a7–12. There is of course no reason to assume that in Cael. 300a7–12 too Aristotle conceives Plato's fundamental bodies as 'stacks' of planes – the polyhedral molecules of the four elements consist of planes in the sense that the latter make up the faces of the former: since, however, in Cael. 300a7–12 Aristotle generalizes what holds for bodies and planes to planes and lines as well as to lines and points, planes can be made from lines and lines from points only if planes are 'bundles' of lines and lines 'strings' of points.

8. The primacy of points, its Academic provenance and its dialectical use by Aristotle

There are some indications that the characterization of points as primary, or by implication as the ultimate elements of the fundamental physical bodies, in Cael. 300a7–12 is of Academic provenance. In Met. 1014b6–9 Aristotle illustrates the definition of element as the fundamental and irreducible constituent of all, most or many things with an apparently Academic view which takes the 'one' and the 'point' to be principles of this sort: since, therefore,

28) For the conception of points, lines and planes as potential divisions of lines, planes and solids respectively see Met. 1002b8–11, 1060b13–15, DA 430b20, Phys. 262a21–25, 263a23–29.

29) See Kouremenos (above, n. 26) 106–107.

in Cael. 299a6–11 and Cael. 300a7–12 he argues in effect against the assumption that points are the irreducible elements of physical reality, he might be assumed to attack in Cael. 299a6–11 and Cael. 300a7–12 a thesis Plato himself is committed to. It is true that, setting out to explain the formation of the polyhedral molecules, Plato hints enigmatically at some principles or elements more fundamental than the triangles in the faces of the molecules (Ti. 53c4–d7) and that he did posit the 'one' as a principle of numbers and geometric magnitudes (Aristotle, Met. 987b14–21): since, therefore, in an argument against the Academic principles of numbers and geometric magnitudes at Met. 1077a32–36 Aristotle objects to the notion that planes, lines and points are the matter of solids, one might be tempted to assume that in the light of Met. 1014b6–9 and 1077a32–36 Aristotle brings out in Cael. 299a6–11 and Cael. 300a7–12 the absurdities implicit in Plato's own views on the ultimate elements of his polyhedral molecules. There is, however, no evidence that points are the ultimate elements of the polyhedral molecules Plato refers to cryptically. That the ultimate elements or principles in Plato's element theory are not the triangles in the faces of the polyhedral molecules of the four elements but rather points can be plausibly construed as Aristotle's own inference.

According to Aristotle Plato defined something as prior to something else κατὰ φύσιν καὶ οὐσίαν if the latter cannot be without the former (Met. 1019a1–4) and in his definition of οὐσία Aristotle attributes to some the view that the plane and the line are οὐσίαι because they are prior to solids and planes respectively (Met. 1017b17–20; 1028b16–18); this view also occurs in Met. 1090b5–11 among various Academic theories on the nature of mathematical objects and was most probably espoused by Plato himself in the light of Aristotle's report in Met. 1019a1–4, all the more since interest in the progression of dimensions is evident in Plato's dialogues[30]. Thus it seems that in his philosophy of mathematics Plato accorded some significance to the logical dependence of solids, planes and lines on planes, lines and points: as planes are logically prior to solids, so are lines to planes and points to lines[31].

30) See Annas (above, n. 16) 59.

31) Cf. Annas (above, n. 16) 59: "Solids are bounded by planes and planes by lines; so there could be a plane without a solid but not a solid without a plane and a line without a plane but not a plane without a line. This asymmetry was found interesting because it suggests that the simpler item is prior to the complex in a special sense".

Plato might have construed this dependence as ontological or it might be Aristotle who thinks that Plato misconstrued logical dependence as ontological. Be that as it may, if the assumption in Cael. 300a7–12 that points are to lines as lines are to planes and planes to solids actually amounts to the ontological priority of the logically prior item, it is Plato's own assumption or is imputed to Plato by Aristotle who can argue that, in the light of this assumption, the ultimate elements in Plato's element theory ought to be simply points. Plato posits that the polyhedral molecules of the four elements are made from, and dissolve into, planes which, being ontologically prior to solids, can only be οὐσίαι in that they are the matter (cf. Met. 1077a32–36) or the elements of the polyhedra, as Plato himself has it. Since, however, for Plato lines are ontologically prior to planes and points to lines, there is no cogent reason why these planes ought not to be made from, and thus dissolve into, lines which by the same reasoning are made from, and thus dissolve into, points: thus points turn out to be not only the elements lines are made from but also the ultimate elements the fundamental physical bodies themselves, and thus all physical bodies, are made from and dissolve into.

If this is so, Cael. 299a2–6, 299a6–11, 299a13–b11 and 300a7–12 bring out the absurd implications of a conclusion which is set out in Cael. 300a7–12 and by Aristotle's lights follows naturally from Plato's own premises, i. e. that Plato ought to posit as the ultimate elements or principles not the triangles in the faces of the polyhedral molecules of the four elements but rather points. Lines, moreover, also ought to consist of points and, since Aristotle does not distinguish between points and indivisible lines, he can argue in Cael. 299a2–6 that Plato's element theory entails the existence of indivisible lines, which engenders insurmountable mathematical difficulties. In Cael. 299a6–11 Aristotle argues that the composition of continua from indivisible points is also shown to be untenable by the first argument in Phys. Z and in Cael. 299a13–b11 he attacks afresh the notion that continua consist of indivisible entities (be they points or indivisible lines) again, as he makes clear, from a physical point of view, which means that he takes the first argument in Phys. Z to be physical in character. His final point against this notion is put forth in Cael. 300a7–12, namely that bodies, planes and lines will all dissolve into what is primary so that there can exist no bodies but only points. As Aristotle puts it in Cael.

298b3–4, natural substances are bodies or depend on bodies and in Cael. 268a4–6 he points out that natural entities are either bodies and magnitudes or have body and magnitude or are principles of those that have body and magnitude. If Simplicius is right in identifying bodies and magnitudes with the four elements and those that have body and magnitude with the living things made out of the four elements (In Cael. 6.33–7.1 [Heiberg]), in Cael. 300a7–12 Aristotle seems to reach the implicit conclusion that Plato's element theory cannot even account for the existence of the four elements, the fundamental physical bodies that are among the primary objects of physics (Cael. 298b1–3).

Thessaloniki Theokritos Kouremenos